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Partitions and Compositions

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Abstract: I shall define the Partition and Composition of a positive integer n in this paper. We shall discuss the way to find the compositions of n and introduce the generating function for partition of n. I shall introduce Ferrer's graph to represent the partition and give some of the remarkable theorems in partition. I shall discuss about the various congruencies given by Srinivasa Ramanujan and present the Rogers-Ramanujan Partition Theorem (Without Proof). Finally, I had given the table for partitions for the first 100 natural numbers.

Partition:

A partition of a positive integer n is a unordered decomposition (division) of n in to any number of positive integral parts. The number of partitions of n is denoted by P(n).

Composition:

A composition of a positive integer n is a ordered decomposition (division) of n in to any number of positive integral parts. The number of compositions of n is denoted by C(n).

For example, the number of partitions of 1 is 1 because 1=1. Similarly the number of partitions of 2 is 2 because 2 = 2, 1+1. The number of partitions of 3 is 3 because 3=3,2+1,1+1+1. The number of partitions of 4 is 5 because
4= 4,3+1,2+2,2+1+1,1+1+1+1. Thus P(1)=1,P(2)=2,P(3)=3,P(4)=5. One can find that P(5)=7,P(6)=11,P(7)=15,P(8)=22,P(9)=30,P(10)=42,P(11)=56,P(12)=77,P(13)=101,…
From the above values we observe that as n increases P(n) increases rapidly.
One of the interesting challenges in discussing partitions is that there is no explicit formula for P(n) in terms of n though there are nice approximations to find P(n) for a given positive integer n. The approximation of P(n) by circle method given by Ramanujan and Hardy is one of the jewels in analytic number theory.

Though there is no exact formula for finding the number of partitions of a given positive integer n there is a simple formula for finding the number of compositions of n.

The number of compositions of 1 is 1 because 1=1. Similarly, the number of compositions of 2 is 2 because 2=2,1+1. The number of compositions of 3 is 4 because 3=3,2+1,1+2,1+1+1.The number of compositions of 4 is 8 because 4=4,3+1,1+3,2+2,2+1+1,1+2+1,1+1+2,1+1+1+1. Thus C(1)=1,C(2)=2,C(3)=4,C(4)=8. One can find that
C(5)=16,C(6)=32,C(7)=64,C(8)=128,C(9)=256,C(10)=512,C(11)=1024,…

From the above values we observe that as n increases C(n) increases rapidly but we find a pattern for the values of C(n) that for each n, C(n) is a power of 2. In fact it is easy to note

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that $C(n)=2^{n-1}$. We also observe another interesting fact that though the number of compositions of n is much larger than the number of partitions of n for n >2 we have a compact formula for C(n) but there is no such formula for P(n). Though there is no exact formula for P(n) there is a technique for obtaining P(n) via generating functions.

Generating Function for P(n):

P(n) is the coefficient of x^n in the product $(1-x)^{-1}.(1-x^2)^{-1}.(1-x^3)^{-1}….(1-x^n)^{-1}…$denoted by $\Pi^{\infty}_{r=1}\ 1/(1-x^r)$. For example, P(4) is the coefficient of x^4 in the product given by

$(1-x)^{-1}.(1-x^2)^{-1}.(1-x^3)^{-1}.(1-x^4)^{-1}…$ =
$(1+x+x^2+x^3+x^4+…)(1+x^2+x^4+x^6+…)(1+x^3+x^6+x^9+…)(1+x^4+x^8+x^{12}+…)$ =
$1+x+2x^2+3x^3+5x^4+…$

The coefficient of x^4 in the above expression is 5. Thus P(4) = 5. Similarly one can find P(n) using the above generating function for other values of n.

Ferrer's diagram for representing partition:

Many theorems about partitions can be proved easily by representing each partition by a diagram of dots, known as a Ferrer's diagram. Here we represent each term of the partition by a row of dots, the terms in descending order, with the largest at the top.
For example, the partition (5,4,2,1) of 12 is represented by the diagram below:

The partition we get by reading the Ferrer's diagram by columns instead of rows is called the conjugate partition of the original partition. So the conjugate partition of the partition (5,4,2,1) of 12 is (4,3,2,2,1) as shown in the above diagram. We observe that the partition of 12 represented by the first graph has 4 parts and the partition of 12 represented by the second graph has 4 as the largest part. We make use of this simple property to establish the following nice result.

1. "If $P_k(n)$ represent the number of partitions of n in to k parts ($1\leq k\leq n$) then the number of partitions of n in to parts the largest of which is k, is $P_k(n)$".

Proof: For each partition for which the largest part is k, the conjugate partition (by Ferrer's graph) has k parts and vice versa. This completes the proof..

For example the number of partitions of 6 in to 3 parts is 3 which are (2,2,2), (1,2,3) and (4,1,1).Hence $P_3(6)=3$. The number of partitions of 6 in to parts for which the largest part is 3 is 3 which are (3,3),(3,2,1) and (3,1,1,1). Here we observe that (2,2,2) and (3,3) are conjugate partitions. (4,1,1) and (3,1,1,1) are conjugate partitions, while (1,2,3) is self conjugate. Now we shall try to establish some of the other results in Partitions.

If $P_d(n)$ represent the number of partitions of n with distinct parts then $P_d(n)$ is the coefficient of x^n in the product $(1+x)(1+x^2)(1+x^3)\ldots(1+x^n)\ldots = \Pi^{\infty}{}_{r=1}\ (1+x^r)$. For example $P_d(4)$ is the coefficient of x^4 in the product
$(1+x)(1+x^2)(1+x^3)(1+ x^4)\ldots = 1+x+ x^2+2\ x^3+2\ x^4+\ldots$ The coefficient of x^4 in this expression is 2. Thus $P_d(4)=2$ which means that 4 has two partitions with distinct parts given by 3+1,4

If $P_o(n)$ represent the number of partitions of n with odd number of parts then $P_o(n)$ is the coefficient of x^n in the expression given by $(1-x)^{-1}.\ (1-x^3)^{-1}.(1-x^5)^{-1}\ldots$
For example $P_o(5)$ is the coefficient of x^5 in the product
$(1-x)^{-1}.\ (1-x^3)^{-1}.(1-x^5)^{-1}\ldots= 1+x+ x^2+2\ x^3+2\ x^4+3\ x^5+\ldots$ The coefficient of x^5 is 3 and thus $P_o(5)$ =3 which means that there are 3 partitions of 5 in to odd parts given by 1+1+1+1+1, 1+1+3,5.

We are now in a position to establish a result proved by Euler two hundred years ago, to a theorem which is today named after him.

2. Euler's Theorem: For any positive integer n, the number of partitions of n with distinct parts equals the number of partitions of n with odd parts.
(That is, $P_d(n)= P_o(n)$.for every positive integer n).

Proof: $P_d(x) = \Pi^{\infty}{}_{r=1}\ (1+x^r)= (1+x)(1+x^2)(1+x^3)(1+x^4)\ldots$
= 1-x2.1-x4.1-x6.1-x8….
1-x 1-x2 1-x3 1-x4….
$= (1-x)^{-1}.\ (1-x^3)^{-1}.(1-x^5)^{-1}\ldots$
$= P_o(x)$.

Therefore for every positive integer n, the coefficient of x^n in $P_d(x)$ and $P_o(x)$ must be equal and this completes the proof.

Congruencies in P(n):

Srinivasa Ramanujan was the first mathematician to observe the congruencies in partitions. MacMahon had calculated a table of P(n) for the first 200 values of n, and from this Ramanujan observed such congruence properties for P(n).
In particular in 1919, Ramanujan proved the following congruencies concerning P(n):

$$P(5n+4)\equiv 0 \pmod 5$$
$$P(7n+5)\equiv 0 \pmod 7$$
$$P(11n+6)\equiv 0 \pmod{11}$$

In otherwords, P(4),P(9),P(14),… are divisible by 5
P(5),P(12),P(19),…are divisible by 7
P(6),P(17),P(28),…are divisible by 11.

For n = 0, these congruencies imply that P(4)=5,P(5)=7,P(6)=11.

Ramanujan also proved congruencies with moduli $5^2, 7^2, 11^2$ given by

$$P(25n+24) \equiv 0 \pmod{5^2}$$
$$P(49n+47) \equiv 0 \pmod{7^2}$$
$$P(121n+116) \equiv 0 \pmod{11^2}$$

Ramanujan gave several other conjectures in partition and with his insight more work has been carried out in the subsequent years in proving his conjectures and proving more results in congruencies in partition.
Rogers(1894) and Ramanujan(1913) independently found a theorem that is much deeper than Euler's theorem, even though it looks almost the same.

3. Rogers-Ramanujan Theorem on partition:

"If odd numbers are characterized by integers with 1,3,5,7or 9 as last digit, and we call integers with last digit 1,4,6or 9 as strange numbers, then the number of partitions of n in to strange parts equals the number of partitions of n in to distinct parts no two of which are consecutive integers".

The following example illustrates the above theorem.

If n=12 then we have nine strange parts given by 11+1,9+1+1+1,6+6,6+4+1+1,6+1+1+1+1+1+1,4+4+4,4+4+1+1+1+1, 4+1+1+1+1+1+1+1+1,1+1+1+1+1+1+1+1+1+1+1+1 and we see that there are nine partitions of 12 in to distinct parts without consecutive integers given by 12,11+1,10+2,9+3,8+4,8+3+1,7+5,7+4+1,6+4+2.

The above theorem remained an open problem for 60 years, until A.Garsia and S.Milne found the proof which they published in a paper entitled :"A Rogers-Ramanujan bijection".

Now I shall present the table of partition of first 100 natural numbers.

n	P(n)
1	1
2	2
3	3
4	5
5	7
6	11
7	15
8	22
9	30
10	42
11	56
12	77
13	101

14	135
15	176
16	231
17	297
18	385
19	490
20	627
21	792
22	1002
23	1255
24	1575
25	1958
26	2436
27	3010
28	3718
29	4565
30	5604
31	6842
32	8349
33	10143
34	12310
35	14883
36	17977
37	21637
38	26015
39	31185
40	37338
41	44583
42	53174
43	63261
44	75175
45	89134
46	105558
47	124754
48	147273
49	173525
50	204226
51	239943
52	281589
53	329931
54	386155
55	451276
56	526823
57	614154
58	715220
59	831820

60	966467
61	1121505
62	1300156
63	1505499
64	1741630
65	2012558
66	2323520
67	2679689
68	3087735
69	3554345
70	4087968
71	4697205
72	5392783
73	6185689
74	7089500
75	8118264
76	9289091
77	10619863
78	12132164
79	13848650
80	15796476
81	18004327
82	20506255
83	23338469
84	26543660
85	30167357
86	34262962
87	38887673
88	44108109
89	49995925
90	56634173
91	64112359
92	72533807
93	82010177
94	92669720
95	104651419
96	118114304
97	133230930
98	150198136
99	169229875
100	190569292.

Remarks: I present this paper as a tribute to the greatest mathematical genius of India, Srinivasa Ramanujan.

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